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MIXED WIDTH-INTEGRALS OF CONVEX BODIES

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ABSTRACT
The mixed width-integrals are defined and shown to have properties similar to those of the mixed volumes of Minkowski. An inequality is established for the mixed width-integrals analogous to the Fenchel–Aleksandrov inequality for the mixed volumes. An isoperimetric inequality (involving the mixed width-integrals) is presented which generalizes an inequality recently obtained by Chakerian and Heil. Strengthened versions of this general inequality are obtained by introducing indexed mixed width-integrals. This leads to an isoperimetric inequality similar to Busemann's inequality involving concurrent cross-sections of convex bodies.

In recent papers [4, 5] Chakerian proved that if $K_1, \cdots, K_n$ are rotations of a fixed convex body (compact convex set with non-empty interior) in Euclidean n-space and $b_{K_i}(u)$ is half the width of K_i in the direction u, then

$$V(K_1)\cdots V(K_n) \leqq \left[\frac{1}{n}\int_\Omega b_{K_1}(u)\cdots b_{K_n}(u)\,dS(u)\right]^n$$

with equality if and only if the K_i are n-balls. In the inequality above, $V(K_i)$ denotes the n-dimensional volume of K_i, Ω denotes the surface of the unit n-ball, and dS denotes the area element on Ω. For plane convex bodies, this inequality was proven by Heil [9], generalizing a result of Radziszewski [14] which was later rediscovered by Chernoff [6].

In this paper we show that the inequality established by Chakerian is a general inequality which holds for arbitrary convex bodies K_i (which need have no relation to one another). By considering power-means, we obtain strengthened versions of this general inequality. This, in turn, leads to an isoperimetric inequality similar to an inequality of Busemann [3] involving concurrent cross-sections of convex bodies.

The setting for this paper is Euclidean n-dimensional space, R^n. We shall use $\mathcal{K}^n$ to denote the space of convex bodies, endowed with the Hausdorff topology.

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The letter K (with subscripts) will be used to denote convex bodies, exclusively. The volume of the unit n-ball, U, will be denoted by ω_n. Convex bodies $K_1, \cdots, K_r$ are said to have *similar width* if there exist constants $\lambda_1, \cdots, \lambda_r > 0$ such that $\lambda_1 b_{K_1}(u) = \cdots = \lambda_r b_{K_r}(u)$ for all $u \in \Omega$; they are said to have *constant width jointly* if the product $b_{K_1}(u) \cdots b_{K_r}(u)$ is constant for all $u \in \Omega$. For reference see Bonnesen and Fenchel [1] or Hadwiger [7].

Inspired by Chakerian's inequality, we define the mixed width-integral

$$A(K_1, \cdots, K_n) = \frac{1}{n} \int_\Omega b_{K_1}(u) \cdots b_{K_n}(u) dS(u).$$

By this definition, A is a map

$$A : \underbrace{\mathcal{K}^n \times \cdots \times \mathcal{K}^n}_{n} \to R.$$

It is positive, continuous, translation invariant, monotone under set inclusion, and homogeneous of degree one in each variable.

Just as the cross-sectional measures $W_i(K)$ are defined to be the special mixed volumes

$$V(\underbrace{K, \cdots, K}_{n-i}, \underbrace{U, \cdots, U}_{i}),$$

the width-integrals $B_i(K)$ can be defined as the special mixed width-integrals

$$A(\underbrace{K, \cdots, K}_{n-i}, \underbrace{U, \cdots, U}_{i}).$$

It was shown in [11] that the width-integrals have a great many properties in common with the cross-sectional measures. Similarly, the mixed width-integrals have many properties in common with the mixed volumes.

The following general inequality between mixed volumes is due to Aleksandrov [2, p. 50]:

$$\prod_{i=0}^{m-1} V(K_1, \cdots, K_{n-m}, K_{n-i}, \cdots, K_{n-i}) \leqq V^m(K_1, \cdots, K_n) \qquad [1 < m \leqq n].$$

To establish a similar inequality between the mixed width-integrals, we require the following simple extension of Hölder's inequality:

LEMMA 1. *If $f_0, f_1, \cdots, f_m$ are (strictly) positive continuous functions defined on Ω and $\alpha_1, \cdots, \alpha_m$ are positive constants the sum of whose reciprocals is unity, then*

$$\int_{\Omega} f_0(u) f_1(u) \cdots f_m(u)\, dS(u) \leqq \prod_{i=1}^{m} \left[\int_{\Omega} f_0(u) f_i^{\alpha_i}(u)\, dS(u) \right]^{1/\alpha_i}$$

with equality if and only if there exist positive constants $\lambda_1, \cdots, \lambda_m$ *such that* $\lambda_1 f_1^{\alpha_1}(u) = \cdots = \lambda_m f_m^{\alpha_m}(u)$ *for all* $u \in \Omega$.

Lemma 1 leads to a simple proof (see [10]) of

THEOREM 1.

$$A^m (K_1, \cdots, K_n) \leqq \prod_{i=0}^{m-1} A(K_1, \cdots, K_{n-m}, K_{n-i}, \cdots, K_{n-i}) \qquad [1 < m \leqq n]$$

with equality if and only if $K_{n-m+1}, K_{n-m+2}, \cdots, K_n$ *are all of similar width.*

The general form of Chakerian's inequality that we shall prove is:

THEOREM 2. $V(K_1) \cdots V(K_n) \leqq A^n(K_1, \cdots, K_n)$ $\qquad [K_i \in \mathcal{K}^n]$
with equality if and only if the K_i *are n-balls.*

PROOF. From Jensen's inequality [8, p. 144] we have:

$$n\omega_n^2 \left[\int_{\Omega} b_{K_1}^{-1}(u) \cdots b_{K_n}^{-1}(u)\, dS(u) \right]^{-1} \leqq \frac{1}{n} \int_{\Omega} b_{K_1}(u) \cdots b_{K_n}(u)\, dS(u),$$

with equality if and only if the K_i have constant width jointly. Hölder's inequality [8, p. 140] yields:

$$\prod_{i=1}^{n} \left[\int_{\Omega} b_{K_i}^{-n}(u)\, dS(u) \right]^{-1} \leqq \left[\int_{\Omega} b_{K_1}^{-1}(u) \cdots b_{K_n}^{-1}(u)\, dS(u) \right]^{-n}$$

with equality if and only if the K_i have similar width. In [12] it was shown that

$$V(K_i) \leqq \left[\frac{1}{n\omega_n^2} \int_{\Omega} b_{K_i}^{-n}(u)\, dS(u) \right]^{-1}$$

with equality if and only if K_i is an n-dimensional ellipsoid. By combining these inequalities we obtain the desired result.

A strengthened version of this inequality can be obtained by introducing the concept of mixed width-integrals of order p. For a real number $p \neq 0$ we define the mixed width-integral of order p by

$$A_p(K_1, \cdots, K_n) = \omega_n \left[\frac{1}{n\omega_n} \int_{\Omega} b_{K_1}^{p}(u) \cdots b_{K_n}^{p}(u)\, dS(u) \right]^{1/p}.$$

For p equal to $-\infty$, 0, or ∞ we define the mixed width-integral of order p by

$$A_p(K_1, \cdots, K_n) = \lim_{s \to p} A_s(K_1, \cdots, K_n).$$

As a direct consequence of Jensen's inequality we have:

PROPOSITION.

$$A_p(K_1, \cdots, K_n) \leqq A_q(K_1, \cdots, K_n) \qquad [-\infty \leqq p < q \leqq \infty,\ K_i \in \mathcal{K}^n]$$

with equality if and only if the K_i *have constant width jointly.*

Contained in the proof of Theorem 2 is a proof of the following:

LEMMA 2. $V(K_1) \cdots V(K_n) \leqq A_{-1}^n(K_1, \cdots, K_n) \qquad [K_i \in \mathcal{K}^n]$
with equality if and only if the K_i *are homothetic ellipsoids.*

By combining Lemma 2 with the Proposition we obtain

THEOREM 3.

$$V(K_1) \cdots V(K_n) \leqq A_p^n(K_1, \cdots, K_n) \qquad [-1 < p \leqq \infty, K_i \in \mathcal{K}^n]$$

with equality if and only if the K_i *are n-balls.*

Clearly, Theorem 2 is the special case $p = 1$ of Theorem 3. It follows from the Proposition that for all p such that $-1 < p < 1$ the inequalities of Theorem 3 are stronger than the inequality of Theorem 2. Theorem 3 may also be considered a generalization of Theorem 2 of [12].

If we consider the general inequality

$$V(K_1) \cdots V(K_n) \leqq A_p^n(K_1, \cdots, K_n),$$

then it follows that it holds for p if and only if $-1 \leqq p \leqq \infty$. To see that the inequality cannot hold for any $p < -1$ we merely set all the K_i equal to some fixed (non-spherical) ellipsoid.

Lemma 2 leads to an isoperimetric inequality similar to Busemann's inequality [3, p. 2] relating the volumes and cross-sections of convex bodies.

For a convex body K and a fixed direction $u \in \Omega$, let $\alpha_K(u)$ denote the (integralgeometric) mean of the $(n-1)$-dimensional volumes of the intersections of K with the hyperplanes orthogonal to u that pass through the interior of K. For the unit n-ball we have $\alpha_U(u) = \omega_n/2$ for all $u \in \Omega$.

As a direct consequence of our definitions we have:

LEMMA 3. $2\alpha_K(u) b_K(u) = V(K) \qquad [u \in \Omega,\ K \in \mathcal{K}^n].$

If we combine Lemma 2 and Lemma 3 we obtain

THEOREM 4.

$$\frac{2^n}{n\omega_n^2}\int_\Omega \alpha_{K_1}(u)\cdots\alpha_{K_n}(u)\,dS(u) \leqq V(K_1)^{(n-1)/n}\cdots V(K_n)^{(n-1)/n} \qquad [K_i \in \mathcal{K}^n]$$

with equality if and only if the K_i are homothetic ellipsoids.

We note that Theorem 4 may also be regarded as a generalization of an isoperimetric inequality in [13].

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